Semistable Representations of Quivers

Ana Bălibanu

Let Q be a finite quiver with no oriented cycles, I its set of vertices, k an algebraically closed field, and $\mathcal{M}od_kQ$ the category of finite-dimensional representations of Q. A representation of Q is a collection (X_i, φ_α) of vector spaces X_i, one for each vertex $i \in I$, and of homomorphisms $\varphi_\alpha : X_i \to X_j$, one for each arrow $\alpha : i \to j$.

The *dimension type* of $X = (X_i, \varphi_\alpha) \in \mathcal{M}od_kQ$ is

$$\underline{\dim} X = \sum_{i \in I} \dim(X_i) i,$$

and is an element of the free abelian group $\mathbb{Z}I$. The *dimension* of X is

$$\dim(X) = \sum_{i \in I} \dim(X_i),$$

and we can view dim as a element of $(\mathbb{Z}I)^*$.

Fix once and for all a linear map $\theta = \sum_{i \in I} \theta_i i^* \in (\mathbb{Z}I)^*$, and define the *slope* of a non-zero representation X to be

$$\mu(X) = \frac{\theta(X)}{\dim(X)}$$

where by $\theta(X)$ we mean θ applied to the dimension type of X.

Definition. A representation $X \in \mathcal{M}od_kQ$ is *semistable* if for any nonzero submodule $M \leq X$, $\mu(M) \leq \mu(X)$. It is *stable* if whenever M is a proper submodule, $\mu(M) < \mu(X)$. We will say X is μ_0-*(semi)stable* if it is (semi)stable and $\mu(X) = \mu_0$.

1 The Representation Space of a Quiver

Fix a dimension type $d = \sum_{i \in I} d_i i$, and define

$$\mathcal{R}_d = \bigoplus_{\alpha : i \to j} \mathrm{Hom}_k(X_i, X_j),$$

where each X_i is a vector space over k of dimension d_i. $\mathcal{R}_d$ parametrizes the representations of Q of dimension type d, since once we have fixed the vector spaces X_i, any representation is determined precisely by a choice of homomorphisms φ_α corresponding to the arrows. Define

$$G_d = \prod_{i \in I} GL_k(X_i),$$

so that G_d acts on $\mathcal{R}_d$ via

$$(g_i)_{i \in I} \cdot (\varphi_\alpha)_{\alpha : i \to j} = (g_j \varphi_\alpha g_i^{-1})_{\alpha : i \to j}$$

The orbits of G_d on $\mathcal{R}_d$ are in bijective correspondence with the isomorphism classes of Q-representations of dimension type d.

Let $\mu_0 = \frac{\theta(d)}{\dim(d)}$, and define the corresponding character χ_0 of G_d by

$$\chi_0 = \prod_{i \in I} \det(g_i)^{\mu_0 - \theta_i}$$

where we recall that the θ_i's are the coefficients of θ in $(\mathbb{Z}I)^*$.

Theorem 1.1. *A representation $X \in \mathcal{M}od_k Q$ of dimension type d is μ_0-(semi)stable if and only if, as a point in $\mathcal{R}_d$, it is χ_0-(semi)stable in the sense of Mumford's GIT.*

The proof of this theorem will follow [3] and will make use of the Hilbert-Mumford Numerical Criterion, which we recall here:

Theorem. *Let G be a reductive algebraic group acting on a vector space R over k, and let χ be a character of G. Then $x \in R$ is χ-semistable if and only if, for every one-parameter subgroup $\lambda : G \to k^\times$ such that $lim_{t \to 0} \lambda(t) \cdot x$ exists, $\langle \chi, \lambda \rangle \geq 0$. A point $x \in R$ is χ-stable if and only if, for every such one-parameter subgroup $\lambda \neq 0$, $\langle \chi, \lambda \rangle > 0$.*

Proof. (of Theorem 1.1) In order to apply Hilbert-Mumford, we first need to establish how one-parameter subgroups of G_d act on the representation space $\mathcal{R}_d$. Suppose $\lambda : k^\times \to G_d$ is a one-parameter subgroup—then λ induces a (commutative) one-parameter subgroup in each $GL_k(X_i)$ which acts diagonally on X_i by powers of t. Each X_i decomposes into a sum of weight spaces for the action of λ,

$$X_i = \bigoplus_{n \in \mathbb{Z}} X_i^{(n)},$$

where $\lambda(t) \cdot v = t^n v$ for every $v \in X_i^{(n)}$. We will write $X_i^{\geq n} = \bigoplus_{m \geq n} X_i^{(m)}$.

Then each homomorphism $\varphi_\alpha : X_i \to X_j$ decomposes into components

$$\varphi_\alpha^{(mn)} : X_i^{(n)} \to X_j^{(m)},$$

and λ acts on each component by

$$\begin{aligned}
\lambda(t) \cdot \varphi_\alpha^{(mn)} &= \lambda(t)|_{X_j} \cdot \varphi_\alpha^{(mn)} \cdot \lambda(t)|_{X_i}^{-1} \\
&= t^m \cdot \varphi_\alpha^{(mn)} \cdot t^{-n} \\
&= t^{m-n} \varphi_\alpha^{(mn)}
\end{aligned}$$

Hence if the limit of $\lambda(t) \cdot (\varphi_\alpha)_{\alpha:i\to j}$ is to exist as $t \to 0$, we must have $\varphi_\alpha^{(mn)} = 0$ whenever $m < n$ and for every arrow α. Then

$$\varphi_{\alpha:i\to j} : X_i^{\geq n} \to X_j^{\geq n}$$

and each $M^{\geq n} = (X_i^{\geq n}, \varphi_\alpha)$ is a subrepresentation of $X = (X_i, \varphi_\alpha)$. This makes

$$\ldots \subseteq M^{\geq n+1} \subseteq M^{\geq n} \subseteq M^{\geq n-1} \subseteq \ldots$$

a filtration of X by subrepresentations. Because X is finite-dimensional, the filtration is finite.

So we have seen that one-parameter subgroups for which the limit exists induce filtrations of X by subrepresentations. In fact, whenever we have a filtration

$$\ldots \subseteq M^{\geq n+1} \subseteq M^{\geq n} \subseteq M^{\geq n-1} \subseteq \ldots$$

of X by subreps, we can extract (non-uniquely) a one-parameter subgroup λ by declaring that $\lambda(t)$ acts on a complement of $M_i^{\geq n+1}$ in $M_i^{\geq n}$ by t^n.

Now, suppose that X is μ_0-semistable, and suppose $\lambda : k^\times \to G_d$ is a one-parameter subgroup such that $\lim_{t\to 0} \lambda(t) \cdot X$ exists. Then λ induces a filtration of X by subrepresentations as above, and we have

$$\begin{aligned}
\chi_0(\lambda(t)) &= \prod_{i\in I} \det(\lambda(t)_i)^{\mu_0 - \theta_i} \\
&= \prod_{i\in I} \prod_{n\in\mathbb{Z}} \det(\lambda(t)|_{X_i^{(n)}})^{\mu_0 - \theta_i} \\
&= \prod_{i\in I} \prod_{n\in\mathbb{Z}} (t^n)^{\dim(X_i^{(n)})(\mu_0 - \theta_i)}
\end{aligned}$$

which gives

$$\begin{aligned}
\langle \chi_0, \lambda \rangle &= \sum_{i\in I} \sum_{n\in\mathbb{Z}} n \cdot \dim(X_i^{(n)})(\mu_0 - \theta_i) \\
&= \sum_{n\in\mathbb{Z}} n \Big(\dim(M^{\geq n}/M^{\geq n+1})\mu_0 - \theta(M^{\geq n}/M^{\geq n+1}) \Big) \\
&= \sum_{n\in\mathbb{Z}} n \cdot \dim(M^{\geq n})\mu_0 - n \cdot \dim(M^{\geq n+1})\mu_0 - n \cdot \theta(M^{\geq n}) + n \cdot \theta(M^{\geq n+1}) \\
&= \sum_{n\in\mathbb{Z}} \dim(M^{\geq n})\mu_0 - \theta(M^{\geq n})
\end{aligned}$$

Since X is μ_0-semistable and each $M^{\geq n}$ is a subrepresentation,

$$\frac{\theta(M^{\geq n})}{\dim(M^{\geq n})} = \mu(M^{\geq n}) \;\leq\; \mu(X) = \mu_0$$
$$\Rightarrow \dim(M^{\geq n})\mu_0 - \theta(M^{\geq n}) \geq 0$$

and so each term of the sum above is positive. Thus $\langle \chi_0, \lambda \rangle \geq 0$, and by Hilbert-Mumford X is χ_0-semistable.

For the converse, assume X is χ_0-semistable, and first let $\lambda : k^\times \to G_d$ be the trivial one-parameter subgroup. This induces the trivial filtration

$$0 \subset X$$

and since

$$\chi_0(\lambda(t)) = \prod_{i\in I} \det(\lambda(t)_i)^{\mu_0 - \theta_i} = \prod_{i\in I} 1^{\mu_0-\theta_i} = 1$$

we have that

$$0 = \langle \chi_0, \lambda \rangle = \dim(0)\mu_0 - \theta(0) + \dim(X)\mu_0 - \theta(X) = \dim(X)\mu_0 - \theta(X)$$

and we see that $\mu(X) = \mu_0$. Now, let M be a non-zero proper submodule of X. Then the filtration

$$0 \subset M \subset X$$

corresponds to a one-parameter subgroup λ and, by χ_0-semistability and Hilbert-Mumford,

$$\begin{aligned}
0 \leq \langle \chi_0, \lambda \rangle &= \dim(0)\mu_0 - \theta(0) + \dim(M)\mu_0 - \theta(M) + \dim(X)\mu_0 - \theta(X) \\
&= \dim(M)\mu_0 - \theta(M)
\end{aligned}$$

so

$$\mu(M) = \frac{\theta(M)}{\dim(M)} \le \mu_0 = \mu(X)$$

and thus X is μ_0-semistable.

That X is μ_0-stable if and only if it is χ_0-stable follows from exactly the same arguments by replacing each inequality by a strict inequality. □

2 The Harder-Narasimhan Filtration

Now we pivot to a more category-theoretic perspective. The following results, which are proved for representations of Q, hold with little modification in any abelian category if θ and dim are chosen correctly as positive functions on the Grothendieck group.[2]

Lemma 2.1. *If*

$$0 \longrightarrow M \longrightarrow X \longrightarrow N \longrightarrow 0$$

is a short exact sequence in $\mathcal{M}od_k Q$, then

$$\mu(M) \le \mu(X) \iff \mu(X) \le \mu(N) \iff \mu(M) \le \mu(N).$$

Proof. Note that since θ and dim are defined only on dimension types, we have $\theta(X) = \theta(M) + \theta(N)$ and $\dim(X) = \dim(M) + \dim(N)$.

$$\begin{aligned}
\frac{\theta(M)}{\dim(M)} &\le \frac{\theta(M)+\theta(N)}{\dim(M)+\dim(N)} \\
&\iff \theta(M)\dim(M) + \theta(M)\dim(N) \le \theta(M)\dim(M) + \theta(N)\dim(M) \\
&\iff \theta(M)\dim(N) \le \theta(N)\dim(M) \\
&\iff \frac{\theta(M)}{\dim(M)} \le \frac{\theta(N)}{\dim(N)}
\end{aligned}$$

$$\begin{aligned}
\frac{\theta(M)+\theta(N)}{\dim(M)+\dim(N)} &\le \frac{\theta(N)}{\dim(N)} \\
&\iff \theta(N)\dim(N) + \theta(M)\dim(N) \le \theta(N)\dim(N) + \theta(N)\dim(M) \\
&\iff \theta(M)\dim(N) \le \theta(N)\dim(M) \\
&\iff \frac{\theta(M)}{\dim(M)} \le \frac{\theta(N)}{\dim(N)}
\end{aligned}$$

□

This lemma is referred to as the "seesaw property" in [2], and will be very useful to us in what follows. In particular, it tells us that when X is semistable, the slopes in any short exact sequence are increasing.

Definition. A *Harder-Narasimhan (HN) filtration* of $X \in \mathcal{M}od_k Q$ is a filtration by subrepresentations

$$0 \subset X_1 \subset X_2 \subset \ldots \subset X_s = X$$

such that

- X_k/X_{k-1} is semistable
- $\mu(X_1) > \mu(X_2/X_1) > \ldots > \mu(X_s/X_{s-1})$

Theorem 2.2. *Every $X \in \mathcal{M}od_k Q$ has a unique HN-filtration.*

We will first require an important lemma.

Lemma 2.3. *Among the subrepresentations of $X \in \mathcal{M}od_k Q$ of maximal slope, there is a unique one of maximal dimension.*

Proof. Towards a contradiction, suppose that M and N are two distinct subrepresentations of X of maximal slope and the same (maximal) dimension. Then $M + N$ is a subrepresentation of X of strictly greater dimension.

We have the short exact sequences

$$0 \longrightarrow M \longrightarrow M + N \longrightarrow \frac{N}{M \cap N} \longrightarrow 0$$

$$0 \longrightarrow M \cap N \longrightarrow N \longrightarrow \frac{N}{M \cap N} \longrightarrow 0$$

Because of Lemma 2.1, the slopes in the first sequence are decreasing by the maximality of $\mu(M)$, and the slopes in the second are increasing by the maximality of $\mu(N)$, so that

$$\mu(M) \geq \mu\Big(\frac{N}{M \cap N}\Big) \geq \mu(N)$$

and thus equality holds everywhere in both exact sequences. In particular $\mu(M) = \mu(M+N)$, and $M+N$ is a subrepresentation of maximal slope and greater dimension than either M or N—a contradiction. □

Proof. (of Theorem 2.2) We will proceed by induction on the dimension of X. Choose X_{max} to be the unique submodule of X of maximal slope and maximal dimension among these. The quotient X/X_{max} has strictly smaller dimension than X, and so it has a unique HN-filtration by the inductive hypothesis. Since subrepresentations of X/X_{max} correspond precisely to subrepresentations of X containing X_{max}, we will write the HN-filtration on X/X_{max} suggestively as

$$0 \subset X_2/X_{max} \subset \ldots \subset X_s/X_{max} = X/X_{max}.$$

Under the quotient $X \to X/X_{max}$, this filtration pulls back to the filtration by subrepresentations

$$0 \subset X_{max} \subset X_2 \subset \ldots \subset X_s = X$$

on X. Since the quotients of this new filtration are the same as that of the filtration on X/X_{max},

$$\frac{X_k}{X_{k-1}} \cong \frac{X_k/X_{max}}{X_{k-1}/X_{max}},$$

they satisfy the properties required by the definition of the HN-filtration. The only thing to check is that X_{max} itself is semistable and has greater slope than the subsequent quotients.

That it is semistable is clear since any submodule of X_{max} is also a submodule of X and thus has smaller slope than X_{max}. Moreover, we have the short exact sequence

$$0 \longrightarrow X_{max} \longrightarrow X_2 \longrightarrow X_2/X_{max} \longrightarrow 0$$

in which the slopes are decreasing by the maximality of the slope of X_{max}, so we have $\mu(X_{max}) \geq \mu(X_2/X_{max})$. Equality cannot hold, since then we would have $\mu(X_{max}) = \mu(X_2)$ and $X_{max} \subset X_2$, contradicting the maximality of the dimension of X_{max}. Consequently

- X_{max} and X_k/X_{k-1} are semistable
- $\mu(X_{max}) > \mu(X_2/X_{max}) > \ldots > \mu(X_s/X_{s-1})$

and our filtration is indeed a Harder-Narasimhan filtration.

It remains to show uniqueness. It will follow immediately from the uniqueness of the HN filtration on X/X_{max} once we have shown that X_{max} must be the first subrepresentation in any HN-filtration of X.

Suppose

$$0 \subset X_1 \subset X_2 \subset \ldots \subset X_s = X$$

is a HN-filtration on X. Since X_{max} is uniquely determined by the requirement that it have maximal dimension among the subrepresentations of maximal slope, it suffices to check that X_1 also has this property.

Suppose towards a contradiction that X_1 does not have maximal slope, and let k be the smallest index such that there is a subrepresentation $Y \subset X_k$ with $\mu(Y) > \mu(X_1)$. Since X_1 is semistable, $k \neq 1$. Then we have the short exact sequence

$$0 \longrightarrow X_{k-1} \cap Y \longrightarrow Y \longrightarrow \frac{Y}{X_{k-1} \cap Y} \cong \frac{X_{k-1} + Y}{X_{k-1}} \longrightarrow 0$$

Now, $(X_{k-1} + Y)/X_{k-1}$ is a subrepresentation of X_k/X_{k-1}, which is semistable, so

$$\mu\Big(\frac{X_{k-1} + Y}{X_{k-1}}\Big) \leq \mu\Big(\frac{X_k}{X_{k-1}}\Big) < \mu(X_1) < \mu(Y).$$

If $X_{k-1} \cap Y = 0$, we have an isomorphism $Y \cong (X_{k-1} + Y)/X_{k-1}$ that gives us $\mu(Y) = \mu((X_{k-1} + Y)/X_{k-1})$, which contradicts the strict inequalities above.

If $X_{k-1} \cap Y \neq 0$, the inequality tells us that the slopes in our short exact sequence are decreasing, so

$$\mu(X_{k-1} \cap Y) \geq \mu(Y) > \mu(X_1),$$

and so X_{k-1} contains the subrepresentation $X_{k-1} \cap Y$ of slope strictly greater than that of X_1, contradicting the minimality of k.

Now suppose, again towards a contradiction, that X_1 does not have maximal dimension among the subrepresentations of maximal slope. Let k be the smallest index such that there is a subrepresentation $Y \subset X_k$ with $\mu(Y) = \mu(X_1)$ and $dim(Y) > dim(X_1)$. Once again we inspect the short exact sequence

$$0 \longrightarrow X_{k-1} \cap Y \longrightarrow Y \longrightarrow \frac{Y}{X_{k-1} \cap Y} \cong \frac{X_{k-1} + Y}{X_{k-1}} \longrightarrow 0$$

in which the slopes are increasing since Y, having maximal slope, is semistable. But then

$$\mu(Y) \leq \mu\Big(\frac{X_{k-1} + Y}{X_{k-1}}\Big) \leq \mu\Big(\frac{X_k}{X_{k-1}}\Big) < \mu(X_1)$$

where the second inequality follows from the semistability of X_k/X_{k-1} and the third inequality follows from the properties of the HN-filtration. Since this third inequality is required to be strict by the HN hypotheses, and since we assumed $\mu(Y) = \mu(X_1)$, we have reached a contradiction.

So X_1 must have maximal dimension among the subrepresentations of maximal slope, so by Lemma 2.3, $X_1 = X_{max}$. Then uniqueness of the HN-filtration on X follows inductively from the uniqueness of the HN-filtration on X/X_{max}. □

3 The HN-Stratification

The existence and uniqueness of HN-filtrations will allow us to stratify the points of $\mathcal{R}_d$ according to the dimension types of their HN-filtrations, following [1].

Definition. If $X \in \mathcal{M}od_k Q$ is a representation with HN-filtration

$$0 \subset X_1 \subset X_2 \subset \ldots \subset X_s = X,$$

the *HN type* of X is the tuple of dimension types

$$d^* = \left(\underline{\dim}(X_1), \underline{\dim}\left(\frac{X_2}{X_1}\right), \ldots, \underline{\dim}\left(\frac{X_s}{X_{s-1}}\right)\right)$$

We will refer to a dimension type d as *semistable* if there is a semistable Q-representation of dimension type d. A tuple of dimension types $d^* = (d^1, d^2, \ldots, d^s)$ will be called of *HN type* if

· each d^k is semistable

· $\mu(d^1) > \mu(d^2) > \ldots > \mu(d^s)$

Definition. If $d^* = (d^1, d^2, \ldots, d^s)$ is a tuple of HN type and $\sum d^k = d$, the *HN-stratum of type* d^* is the subset $\mathcal{R}^{HN}_{d^*} \subset \mathcal{R}_d$ of representations $X \in \mathcal{R}_d$ whose HN type is d^*.

Theorem 3.1. *$\mathcal{R}^{HN}_{d^*}$ is irreducible and locally closed in $\mathcal{R}_d$.*

Proof. Fix a flag F^* of type d^*:

$$0 \subset F^1 \subset F^2 \subset \ldots \subset F^s = X$$

—that is, each F^k is a collection of vector spaces $\{F^k_i\}$, one for each vertex $i \in I$, such that $\underline{\dim}(F^k/F^{k-1}) = d^k$. For every i,

$$0 \subset F^1_i \subset F^2_i \subset \ldots \subset F^s_i = X_i$$

is a flag of subspaces in X_i.

Let $Z \subset \mathcal{R}_d$ be the subset of representations $X = (X_i, \varphi_\alpha)$ that are compatible with this flag—that is, for which

$$\varphi_\alpha(F^k_i) \subseteq F^k_j$$

for every arrow $\alpha : i \to j$. On such an X, F^* gives a filtration by subrepresentations. The set Z is a closed subset of $\mathcal{R}_d$, and there is a natural projection

$$\pi : Z \longrightarrow \mathcal{R}_{d^1} \times \mathcal{R}_{d^2} \times \ldots \times \mathcal{R}_{d^s}$$

obtained by mapping each compatible representation X to the sequence of quotients $(F^1, F^2/F^1, \ldots, F^s/F^{s-1})$ given by restricting the φ_α's appropriately. The preimage $Z_0 = \pi^{-1}(\mathcal{R}^{ss}_{d^1} \times \mathcal{R}^{ss}_{d^2} \times \ldots \times \mathcal{R}^{ss}_{d^s})$ is an open subset of Z.

Let P_{d^*} be the parabolic subgroup of G_d preserving the flag F^*—that is, P_{d^*} is the product of the parabolic subgroups $P_i \subset GL(X_i)$ preserving the flags

$$0 \subset F^1_i \subset \ldots \subset F^s_i = X_i.$$

Then P_{d^*} acts on Z and, since the group action preserves semistability, it also acts on Z_0.

We consider the fiber bundle

$$\begin{aligned} m :\ & G_d \times^{P_{d^*}} Z \longrightarrow \mathcal{R}_d \\ & (g, X) \longmapsto g \cdot X \end{aligned}$$

and its restriction to Z_0

$$\begin{aligned} m_0 :\ & G_d \times^{P_{d^*}} Z_0 \longrightarrow \mathcal{R}_d \\ & (g, X) \longmapsto g \cdot X \end{aligned}$$

Because the action of G_d is transitive on flags, any representation with some filtration of type d^* is G_d-conjugate to one in which the vector spaces of the filtration are given by F^*. Thus, the image of m is the subset $\mathcal{R}_{d^*} \subset \mathcal{R}_d$ of representations that have some filtration of type d^*. Since $\mathcal{R}_{d^*}$ is closed in $\mathcal{R}_d$, m is closed. Since it is a fiber bundle, it is also open.

The image of m_0 is the set of points in $\mathcal{R}_{d^*}$ in which the filtration of type d^* has semistable quotients—so it is precisely the stratum $R_{d^*}^{HN}$. In fact, m_0 is a bijection onto this stratum:

Suppose $m_0(g_1, X_1) = m_0(g_2, X_2)$, so $g_1 \cdot X_1 = g_2 \cdot X_2$. Since $g_2^{-1} g_1 X_1 = X_2 \in Z_0$, the filtration F^* on $g_2^{-1} g_1 X_1$ has semistable quotients. Since $g_2^{-1} g_1 X_1 \in R_{d^*}^{HN}$, the image of the filtration F^* under $g_1^{-1} g_2$ also has semistable quotients. But by the uniqueness of HN-filtrations, F^* and its image under $g_2^{-1} g_1$ must then be the same, so $g_2^{-1} g_1$ preserves F^* and we get $g_2^{-1} g_1 \in P_{d^*}$. Then

$$(g_2, X_2) = (g_2, g_2^{-1} g_1 X_1) \sim (g_2 g_2^{-1} g_1, X_1) = (g_1, X_1)$$

so m_0 is injective.

Thus we have an isomorphism

$$m_0 : \ G_d \times^{P_{d^*}} Z_0 \xrightarrow{\sim} \mathcal{R}_{d^*}^{HN}.$$

Since G_d and Z_0 are both irreducible, so is $\mathcal{R}_{d^*}^{HN}$. Since m is both open and closed, and since $G_d \times Z_0$ is open in $G_d \times Z$, $\mathcal{R}_{d^*}^{HN}$ is open in $\mathcal{R}_{d^*}$, and so $\mathcal{R}_{d^*}^{HN}$ is locally closed in $\mathcal{R}_d$. □

We have stratified $\mathcal{R}_d$ into irreducible, locally closed strata, and we have seen that the closure of the HN-stratum $\mathcal{R}_{d^*}^{HN}$ is precisely $\mathcal{R}_{d^*}$. We will see that $\mathcal{R}_{d^*}$ is not always a union of HN-strata, but it is contained in a more or less restricted collection of HN-strata nonetheless. We give a partial ordering on the set of HN-types that will allow us to make this statement more precise.

Definition. For an HN-type $d^* = (d^1, \ldots, d^s)$, define $P(d^*)$ to be the polygon in $\mathbb{N}^2$ with vertices $v_0 = (0,0)$, $v_k = (\sum_{i=1}^{k} \dim(d^i), \sum_{i=1}^{k} \theta(d^i))$. Note that the slope of the edge from v_{k-1} to v_k is

$$\frac{\theta(d^k)}{\dim(d^k)} = \mu(d^k),$$

so the fact that d^* is an HN-type—and in particular that the slopes of the d^k's are decreasing—tells us that $P(d^*)$ is a convex polygon. Now define a partial order $d^* \leq e^*$ if and only if $P(d^*)$ lies on or below $P(e^*)$ in $\mathbb{N}^2$.

Theorem 3.2.

$$\mathcal{R}_{d^*} \subseteq \bigcup_{e^* \geq d^*} \mathcal{R}^{HN}_{e^*}.$$

Proof. Let $X \in \mathcal{R}_{d^*}$. We will show that the HN type e^* of X is greater than or equal to d^*. Since X has a filtration of type d^*, it suffices to show that for an arbitrary subrepresentation $U \subset X$, the point $(dim(U), \theta(U))$ is on or below $P(e^*)$. We will induct on the length of the tuple e^*. If the length is 1, X is semistable and there is nothing to prove.

Let

$$0 \subset X_1 \subset X_2 \subset \ldots \subset X_s = X$$

be the HN-filtration on X, so that

$$0 \subset X_2/X_1 \subset \ldots \subset X_s/X_1 = X/X_1$$

is the HN-filtration on X/X_1, which has length $s-1$. Denote its dimension type f^*. By the induction hypothesis, the point $(dim((U+X_1)/X_1), \theta((U+X_1)/X_1))$ lies on or below $P(f^*)$, so the point $(dim(U+X_1), \theta(U+X_1))$ lies on or below $P(e^*)$, since quotienting by X_1 is the same as translating, in the $\mathbb{N}^2$-plane, by $(-dim(X_1), -\theta(X_1))$.

Now we have the short exact sequences

$$0 \longrightarrow U \longrightarrow U + X_1 \longrightarrow X_1/(U \cap X_1) \longrightarrow 0$$

$$0 \longrightarrow U \cap X_1 \longrightarrow X_1 \longrightarrow X_1/(U \cap X_1) \longrightarrow 0.$$

Since X_1 has maximal slope among the submodules of X, the slopes in the second sequence are increasing and

$$\mu(U + X_1) \leq \mu(X_1) \leq \mu(X_1/(U \cap X_1)).$$

So the slopes in the first sequence are also increasing, so

$$\mu(U) \leq \mu(U + X_1)$$

and since $U + X_1$ is on or below $P(e^*)$ and has a greater x-coordinate than U, U is also on or below $P(e^*)$. □

The inclusion in Theorem 3.2 may be proper. The following example will illustrate this.

Example. Consider the quiver

$$i \longrightarrow j \longrightarrow k$$

and its representations of dimension type $d = i+j+k$ over $\mathbb{C}$. Then a representation will look like

$$\mathbb{C} \xrightarrow{x} \mathbb{C} \xrightarrow{y} \mathbb{C}$$

and

$$\mathcal{R}_d = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C},\mathbb{C}) \times \mathrm{Hom}_{\mathbb{C}}(\mathbb{C},\mathbb{C}) = \mathbb{C} \times \mathbb{C}.$$

Let $\theta = 2i^* + 3j^*$ and take the tuple $d^* = (i, j+k)$. We want to see when X has an HN-filtration of type d^*. Such a filtration will look like

$$0 \,\subset\, \mathbb{C} \xrightarrow{x} 0 \xrightarrow{y} 0 \,\subset\, \mathbb{C} \xrightarrow{x} \mathbb{C} \xrightarrow{y} \mathbb{C},$$

and the first subspace will be a subrepresentation if and only if $x = 0$. Moreover, the quotient

$$0 \xrightarrow{0} \mathbb{C} \xrightarrow{y} \mathbb{C}$$

gives the subrepresentation slopes

$$\begin{aligned} &\mu(0 \xrightarrow{0} \mathbb{C} \xrightarrow{y} \mathbb{C}) = 3/2 \\ &\mu(0 \xrightarrow{0} 0 \xrightarrow{y} \mathbb{C}) = 0 \\ \text{and if } y = 0 & \\ &\mu(0 \xrightarrow{0} \mathbb{C} \xrightarrow{0} 0) = 3. \end{aligned}$$

The last equation makes $0 \xrightarrow{0} \mathbb{C} \xrightarrow{y} \mathbb{C}$ unstable, so we require $y \neq 0$, making our HN-stratum $\mathcal{R}^{HN}_{d^*} = \{0\} \times \mathbb{C}^*$.

Now consider the tuple $e^* = (j, i, k)$, which gives a filtration

$$0 \,\subset\, 0 \xrightarrow{x} \mathbb{C} \xrightarrow{y} 0 \,\subset\, \mathbb{C} \xrightarrow{x} \mathbb{C} \xrightarrow{y} 0 \,\subset\, \mathbb{C} \xrightarrow{x} \mathbb{C} \xrightarrow{y} \mathbb{C}.$$

For the first subspace to be a subrepresentation we need $y = 0$. But in fact that condition is the only one, since the quotients, being one-dimensional, are always semistable. This gives us the stratum $\mathcal{R}^{HN}_{e^*} = \mathbb{C} \times \{0\}$.

Then the closure of $\mathcal{R}^{HN}_{d^*}$ is $\{0\} \times \mathbb{C}$, which is properly contained in the union $\mathcal{R}^{HN}_{d^*} \cup \mathcal{R}^{HN}_{e^*} = \{0\} \times \mathbb{C}^* \cup \mathbb{C} \times \{0\} = \mathbb{C} \times \mathbb{C}$, but is not itself a union of strata.

4 Kempf-Ness and Quivers

From now on we work over $\mathbb{C}$. We recall briefly the theory of Kempf-Ness and its generalization to Mumford's GIT. For this we follow [3], where the proofs of the following statements may also be found.

Let R be a vector space on which a reductive algebraic group G acts, and fix a character χ of G. Let $K \leq G$ be a maximal compact subgroup, $\mathfrak{k}$ its Lie algebra, and recall that $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. Let $(\cdot\,,\cdot)$ be a K-invariant hermitian form on R, and $\eta : R \longrightarrow (i\mathfrak{k})^*$ the moment map, given by

$$\eta_x(A) = (A \cdot x, x)$$

for every $x \in R$ and $A \in i\mathfrak{k}$.

Let $d\chi$ denote the restriction of the derivative of χ to $i\mathfrak{k}$, so that $d\chi \in (i\mathfrak{k})^*$. Then we have the following generalization of Kempf-Ness:

Theorem. *The preimage $\eta^{-1}(d\chi)$ meets each G-orbit in R that is closed in R^{ss}_χ in exactly one K-orbit, and meets no other G-orbits.*

Corollary.

$$\eta^{-1}(d\chi)/K \longleftrightarrow R^{ss}_\chi//(G,\chi)$$

Now, returning to the context of quivers, we have

$$R = \mathcal{R}_d$$
$$G = G_d$$
$$\chi = \chi_0$$

and if we fix a Hermitian inner product on each X_i, invariant under the action of the unitary subgroup $U(d_i) \leq GL(X_i)$, we have the maximal compact subgroup

$$U_d = \prod_{i \in I} U(d_i) \leq G_d.$$

Its lie algebra is $\mathfrak{k} = \bigoplus_{i \in I} \mathrm{Herm}_{\mathbb{C}}(X_i)$, the sum of the spaces of Hermitian endomorphisms on each X_i.

Our Hermitian inner product induces an inner product on each $\mathrm{Hom}_{\mathbb{C}}(X_i, X_j)$ given by

$$(\varphi, \psi) = Tr(\varphi \cdot \psi^*),$$

so the corresponding inner product on $\mathcal{R}_d$ is

$$((\varphi_\alpha), (\psi_\alpha)) = \sum_\alpha Tr(\varphi_\alpha \cdot \psi_\alpha^*).$$

We will compute the moment map with respect to this inner product. Recall that G acts on $\mathcal{R}_d$ by

$$(g_i)_{i\in I} \cdot (\varphi_\alpha)_{\alpha:i\to j} = (g_j \varphi_\alpha g_i^{-1})_{\alpha:i\to j},$$

so $\mathfrak{g}$ acts on $\mathcal{R}_d$ by

$$(A_i)_{i\in I} \cdot (\varphi_\alpha)_{\alpha:i\to j} = (A_j \varphi_\alpha - \varphi_\alpha A_i)_{\alpha:i\to j}.$$

Then the moment map $\eta_{(\varphi_\alpha)}(A_i)$ is

$$\begin{aligned}((A_i)\cdot(\varphi_\alpha), (\varphi_\alpha)) &= ((A_j\varphi_\alpha - \varphi_\alpha A_i), (\varphi_\alpha)) \\ &= \sum_{\alpha:i\to j} Tr(A_j \varphi_\alpha \varphi_\alpha^* - \varphi_\alpha A_i \varphi_\alpha^*) \\ &= \sum_{i\in I} A_i \Big(\sum_{\alpha:\Box\to j} Tr(\varphi_\alpha\varphi_\alpha^*) - \sum_{\alpha:i\to\Box} Tr(\varphi_\alpha^*\varphi_\alpha) \Big)\end{aligned}$$

where the last equality follows from the fact that $Tr(\varphi_\alpha A_i \varphi_\alpha^*) = Tr(A_i \varphi_\alpha^* \varphi_\alpha)$.

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